2006 Mathematics Extension 1 HSC Examination
Terry Lee's Solution

Q1

(a) $\int \frac{dx}{49+x^2} = \frac{1}{7}\tan^{-1}\frac{x}{7} + C$

(b) $\int x^3\sqrt{x^4+8}\,dx = \frac{1}{4}\int \sqrt{u}\,du = \frac{1}{4}\frac{2\sqrt{u^3}}{3} = \frac{1}{6}\sqrt{(x^4+8)^3} + C$

(c) $\lim_{x\to 0}\frac{\sin 5x}{3x} = \lim_{x\to 0}\frac{\sin 5x}{5x}\times\frac{5}{3} = \frac{5}{3}$

(d) $\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta} - 1$

$= \frac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta+\cos\theta} - 1$

$= \sin^2\theta - \sin\theta\cos\theta + \cos^2\theta - 1$

$= 1 - \sin\theta\cos\theta - 1 = -\sin\theta\cos\theta \left(= -\frac{1}{2}\sin 2\theta\right)$

(e) Solving $y = x^3$ and $y = 12x + b$ gives $x^3 - 12x - b = 0$.

Let $f(x) = x^3 - 12x - b$.

$f'(x) = 0, \therefore 3x^2 - 12 = 0, \therefore x^2 = 4, \therefore x = \pm 2.$

$f(2) = 2^3 - 12\times 2 - b = 0, \therefore b = 8 - 24 = -16.$

$f(-2) = (-2)^3 - 12\times(-2) - b = 0, \therefore b = -8 + 24 = 16.$

$\therefore b = \pm 16$

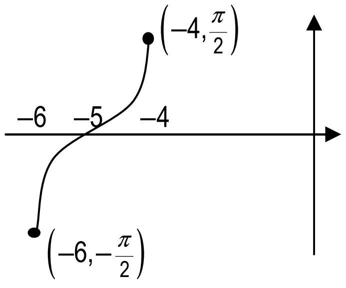
Q2

(a) (i) Domain: $-1 \le x+5 \le 1$ gives $-6 \le x \le -4$.

Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

(ii) $y' = \frac{1}{\sqrt{1-(x+5)^2}}$. When $x = -5, y' = \frac{1}{\sqrt{1}} = 1$.

(iii)



(b) $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n$.

Differentiating both sides with respect to x gives

$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \ldots + n\binom{n}{n}x^{n-1}$.

Substituting x by 2 gives

$n3^{n-1} = \binom{n}{1} + 2\binom{n}{2}2 + 3\binom{n}{3}2^2 + \ldots + n\binom{n}{n}2^{n-1}$.

The rth term is $r\binom{n}{r}2^{r-1}$.

(c) (i) When $x = 0, y = -apr, \therefore U(0, -apr)$

(ii) $y = px - ap^2$ (1)

$y = qx - aq^2$ (2)

(1) − (2) gives

$(p-q)x - a(p^2-q^2) = 0.$

$x = \frac{a(p^2-q^2)}{p-q} = a(p+q).$

Sub. to (1), $y = pa(p+q) - ap^2 = apq$.

$\therefore T(a(p+q), apq)$.

(iii) Since QR is perpendicular to the y-axis, Q and R are symmetrical about the y-axis, $\therefore q = -r$.

$\therefore$ The y-coordinates of U and T are the same.

$\therefore UT$ is perpendicular to the y-axis.

Q3

(a) $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2x}{2}dx$

$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$.

(b) (i) When $x = 1.5$, $f(1.5) = -0.28 < 0$.

When $x = 2$, $f(2) = 0.08 > 0$.

$\therefore$ The curves $\ln x$ and x are continuous on the interval $[1.5, 2]$, $\therefore 3\ln x - x$ is continuous and as it goes from a negative to a positive value it meets the x-axis at least once, $\therefore$ there is (at least) a root between 1.5 and 2.

(ii) $f'(x) = \frac{3}{x} - 1$.

$x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{3\ln 1.5 - 1.5}{\frac{3}{1.5} - 1} = 1.78$ (2 d.p.)

(c) (i) ${}^5P_3 = 60$.

(ii) ${}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5 = 320$.

(d) (i) $\angle QMT + \angle QKT = 180°$.

$\therefore$ QKTM is a cyclic quad (opposite angles are supplementary).

(ii) $\angle KMT = \angle KQT$ (angles subtending the same arc are equal).

(iii) $\angle KQT = \angle PTN$ (the angle between a chord and a tangent is equal to any angle in the alternate segment)

$\therefore \angle KMT = \angle PTN$.

$\therefore$ KM // PT (corresponding angles on parallel lines are equal).

Q4

(a) (i) $\sum\alpha = 1 = -r, \therefore r = -1$.

(ii) $\sum\alpha\beta = \alpha - \alpha - \alpha^2 = s, \therefore s = -\alpha^2$.

$\prod\alpha = -\alpha^2 = -t, \therefore t = \alpha^2$.

$\therefore s + t = 0$

(b) (i) Period = 5 s, $5=\frac{2\pi}{n}, \therefore n=\frac{2\pi}{5}$.

$\therefore$ The equation of motion is $x=18\cos\frac{2\pi}{5}t$, noting that when t = 0, x = 18.

(ii) When x = 9,

$9=18\cos\frac{2\pi}{5}t$

$\cos\frac{2\pi}{5}t=\frac{1}{2}$

$\frac{2\pi}{5}t=\frac{\pi}{3}$

$t=\frac{5}{6}$ s.

(c) (i) $\ddot{x}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=18x^3+27x^2+9x.$

$\frac{1}{2}v^2=\frac{9x^4}{2}+9x^3+\frac{9x^2}{2}+C.$

When $x=-2, v=-6$,

$18=72-72+18+C.$

$C=0.$

$\therefore v^2=9(x^4+2x^3+x^2)=9x^2(x+1)^2$

(ii) $v=\frac{dx}{dt}=-3x(x+1)$ (the minus sign is taken to satisfy that when x = −2, v = −6)

$-3\frac{dt}{dx}=\frac{1}{x(x+1)}$

$-3t=\int\frac{1}{x(x+1)}dx$

(iii) $3t+C=\ln\left(1+\frac{1}{x}\right)$

When $t=0, x=-2, C=\ln\left(1-\frac{1}{2}\right)=\ln\frac{1}{2}.$

$3t=\ln\left(1+\frac{1}{x}\right)-\ln\frac{1}{2}=\ln\left(2+\frac{2}{x}\right).$

$e^{3t}=2+\frac{2}{x}.$

$\frac{2}{x}=e^{3t}-2.$

$x=\frac{2}{e^{3t}-2}.$

Q5

(a) $\frac{dy}{dt}=-7e^{-0.7t}$

But $e^{-0.7t}=\frac{y-3}{10}, \therefore \frac{dy}{dt}=-0.7(y-3).$

(b) $f'(x)=\frac{e^x}{1+e^x}$

As $e^x>0, f'(x)>0, \therefore f(x)$ is monotonic increasing, $\therefore$ f(x) is 1:1, $\therefore$ It has an inverse.

(c) $V=\frac{\pi}{3}\left(3rx^2-x^3\right)$

$\therefore \frac{dV}{dx}=\frac{\pi}{3}\left(6rx-3x^2\right)=\pi\left(2rx-x^2\right).$

$\frac{dV}{dt}=\frac{dV}{dx}\frac{dx}{dt}=\pi\left(2rx-x^2\right)\frac{dx}{dt}=k.$

$\therefore \frac{dx}{dt}=\frac{k}{\pi\left(2rx-x^2\right)}=\frac{k}{\pi x(2r-x)}$

(ii) $\frac{dt}{dx}=\frac{\pi}{k}\left(2rx-x^2\right).$

$t=\frac{\pi}{k}\int\left(2rx-x^2\right)dx.$

Let T_1 be the time to fill up to $\frac{r}{3}$ and T_2 be the time to fill up to $\frac{2r}{3}$.

$T_1=\frac{\pi}{k}\int_0^{\frac{r}{3}}\left(2rx-x^2\right)dx=\frac{\pi}{k}\left[rx^2-\frac{x^3}{3}\right]_0^{\frac{r}{3}}$

$=\frac{\pi}{k}\left(\frac{r^3}{9}-\frac{r^3}{81}\right)=\frac{\pi r^3}{k}\times\frac{8}{81}.$

$T_2=\frac{\pi}{k}\int_0^{\frac{2r}{3}}\left(2rx-x^2\right)dx=\frac{\pi}{k}\left[rx^2-\frac{x^3}{3}\right]_0^{\frac{2r}{3}}$

$=\frac{\pi}{k}\left(\frac{4r^3}{9}-\frac{8r^3}{81}\right)=\frac{\pi r^3}{k}\times\frac{28}{81}.$

$\therefore \frac{T_2}{T_1}=\frac{28}{8}=3.5.$

$\therefore T_2$ is 3.5 times T_1.

(d) (i) $\tan\alpha-\tan\beta=\tan(\alpha-\beta)(1+\tan\alpha\tan\beta)$

$\therefore \tan(n+1)\theta-\tan n\theta=\tan\theta\left(1+\tan(n+1)\theta\tan n\theta\right)$

$\text{RHS}=\frac{1}{\tan\theta}\tan\theta\left(1+\tan(n+1)\theta\tan n\theta\right)$

$=1+\tan(n+1)\theta\tan n\theta$

$=\text{LHS}.$

(ii) When n = 1, LHS = $\tan\theta\tan2\theta$,

From (i), $\tan n\theta\tan(n+1)\theta=$

$-1+\cot\theta\left(\tan(n+1)\theta-\tan n\theta\right)$

$\therefore \tan\theta\tan2\theta=-1+\cot\theta\left(\tan2\theta-\tan\theta\right)$

$=-1+\cot\theta\tan2\theta-1$

$=-2+\cot\theta\tan2\theta=\text{RHS}.$

Assume $\tan\theta\tan2\theta+\tan2\theta\tan3\theta+\ldots$ $+\tan n\theta\tan(n+1)\theta=-(n+1)+\cot\theta\tan(n+1)\theta.$

Required to prove that

$\tan\theta\tan2\theta+\tan2\theta\tan3\theta+\ldots+\tan n\theta\tan(n+1)\theta$

$+\tan(n+1)\theta\tan(n+2)\theta=-(n+2)+\cot\theta\tan(n+2)\theta.$

$\text{LHS} = -(n+1) + \cot\theta\tan(n+1)\theta + \tan(n+1)\theta\tan(n+2)\theta$

$= -(n+1) + \cot\theta\tan(n+1)\theta - 1$

$\quad + \cot\theta\left(\tan(n+2)\theta - \tan(n+1)\theta\right)$

$= -(n+2) + \cot\theta\tan(n+2)\theta$

$= \text{RHS}.$

$\therefore$ The statement is true for $n+1$.

Since the statement is true for $n = 1$, and $n+1$ if true for n, it is true for all $n \ge 1$.

Q6

(a) (i) $L^2 = (Vt\cos\theta - \alpha)^2 + \left(Vt\sin\theta - \frac{1}{2}gt^2 - Vt + \frac{1}{2}gt^2\right)^2$

$= V^2t^2\cos^2\theta - 2\alpha Vt\cos\theta + \alpha^2 + V^2t^2\sin^2\theta + V^2t^2$

$\quad -2V^2t^2\sin\theta$

$= V^2t^2\left(\cos^2\theta + \sin^2\theta\right) + V^2t^2 - 2\alpha Vt\cos\theta + \alpha^2$

$\quad -2V^2t^2\sin\theta$

$= 2V^2t^2 - 2\alpha Vt\cos\theta + \alpha^2 - 2V^2t^2\sin\theta$

$= 2V^2t^2(1-\sin\theta) - 2\alpha Vt\cos\theta + \alpha^2$

(ii) $\frac{dL^2}{dt} = 4V^2t(1-\sin\theta) - 2\alpha V\cos\theta$

$\frac{dL^2}{dt} = 0$ when $t = \frac{2\alpha V\cos\theta}{4V^2(1-\sin\theta)} = \frac{\alpha\cos\theta}{2V(1-\sin\theta)}$

$\frac{d^2L^2}{dt^2} = 4V^2(1-\sin\theta) > 0, \therefore$ The distance is minimum when $t = \frac{\alpha\cos\theta}{2V(1-\sin\theta)}$.

Substituting to (i)

$L^2 = 2V^2\frac{\alpha^2\cos^2\theta}{4V^2(1-\sin\theta)^2}(1-\sin\theta) - \frac{2\alpha^2V\cos^2\theta}{2V(1-\sin\theta)} + \alpha^2$

$= \frac{\alpha^2\cos^2\theta}{2(1-\sin\theta)} - \frac{\alpha^2\cos^2\theta}{1-\sin\theta} + \alpha^2$

$= \alpha^2 - \frac{\alpha^2\cos^2\theta}{2(1-\sin\theta)} = \alpha^2 - \frac{\alpha^2(1-\sin^2\theta)}{2(1-\sin\theta)}$

$= \alpha^2 - \frac{\alpha^2(1+\sin\theta)}{2}$

$= \frac{\alpha^2(2-1-\sin\theta)}{2} = \frac{\alpha^2(1-\sin\theta)}{2}$

$\therefore$ The smallest distance is $\alpha\sqrt{\frac{1-\sin\theta}{2}}$.

(iii) If particle 1 is ascending then $\dot{y} > 0$.

$V\sin\theta - gt > 0$

$V\sin\theta - g\frac{\alpha\cos\theta}{2V(1-\sin\theta)} > 0$

$2V^2\sin\theta(1-\sin\theta) - g\alpha\cos\theta > 0.$

$V^2 > \frac{g\alpha\cos\theta}{2\sin\theta(1-\sin\theta)}.$

$\therefore V > \sqrt{\frac{g\alpha\cos\theta}{2\sin\theta(1-\sin\theta)}}.$

(b) (i) $P(\text{at least 3 not complete}) = P(\text{3 not complete})$

$+P(\text{4 not complete}) = \binom{4}{3}pq^3 + q^4 = 4pq^3 + q^4$

(ii) P(a 4 member team scores point) = 1 – P(at least 3 not complete) $= 1 - 4pq^3 - q^4$

$= 1 - 4(1-q)q^3 - q^4$

$= 1 - 4q^3 + 4q^4 - q^4 = 1 - 4q^3 + 3q^4$

(iii) P(a 2 member team scores point) = 1 – P(both not complete) $= 1 - q^2$.

(iv) $1 - q^2 > 1 - 4q^3 + 3q^4$.

$3q^4 - 4q^3 + q^2 < 0$.

$3q^2 - 4q + 1 < 0$, dividing by q^2,

$(3q-1)(q-1) < 0$

$\frac{1}{3} < q < 1$.

Q7

(a) $A = \frac{1}{2}r^2(2\theta - \sin 2\theta) = \frac{1}{2}\times(2\theta - 2\sin\theta\cos\theta)$

$= r^2(\theta - \sin\theta\cos\theta)$.

(b) $A = \frac{w^2}{4\theta^2}\left(\theta - \frac{1}{2}\sin 2\theta\right)$, since $w = r\times 2\theta, \therefore r = \frac{w}{2\theta}$.

$\frac{dA}{d\theta} = \frac{w^2}{4}\left(\frac{(1-\cos 2\theta)\theta^2 - 2\theta(\theta - \frac{1}{2}\sin 2\theta)}{\theta^4}\right)$

$= \frac{w^2}{4}\left(\frac{(1-\cos 2\theta)\theta - 2\theta + \sin 2\theta}{\theta^3}\right)$

$= \frac{w^2}{4}\left(\frac{-\theta - \theta\cos 2\theta + \sin 2\theta)}{\theta^3}\right)$

$= \frac{w^2}{4}\left(\frac{-\theta - \theta(2\cos^2\theta - 1) + 2\sin\theta\cos\theta)}{\theta^3}\right)$

$= \frac{w^2}{4}\left(\frac{-2\theta\cos^2\theta + 2\sin\theta\cos\theta}{\theta^3}\right)$

$= \frac{w^2\cos\theta}{2}\left(\frac{-\theta\cos\theta + \sin\theta}{\theta^3}\right)$

$= \frac{w^2\cos\theta(\sin\theta - \theta\cos\theta)}{2\theta^3}.$

(c) $g(\theta) = \sin\theta - \theta\cos\theta$.

$g'(\theta) = \cos\theta - (\cos\theta - \theta\sin\theta)$

$\quad = \theta\sin\theta > 0$ for $0 < \theta < \pi$.

$\therefore g(\theta)$ is increasing for $0 < \theta < \pi$.

When $\theta = 0, g(\theta) = 0, \therefore g(\theta) > 0$ for $0 < \theta < \pi$.

(d) $\frac{dA}{d\theta} = 0$ when $\cos\theta = 0$, since $\sin\theta - \theta\cos\theta > 0$

$\therefore \theta = \frac{\pi}{2}$.

$\therefore$ Only 1 value of θ.

(e)

θ	1	$\frac{\pi}{2}$	2
$\cos\theta$	0.54	0	–0.42
$\frac{dA}{d\theta}$	↗	→	↘

$\therefore$ The area of the gutter is maximum when $\theta = \frac{\pi}{2}$.

Maximum area $= \frac{w^2(\pi - 0)}{2\pi^2} = \frac{w^2}{2\pi}$.